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Common errors in Mathematics

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Background

When answering Mathematics questions, students often make errors leading to incorrect answers or the loss of accuracy marks. Many of these errors will be random, occurring through calculation errors or misreading of the question, and will not affect many candidates. However, some errors may be seen in a number of students' scripts. These are sometimes referred to as common errors.

At the end of each examination session, an examiners' report is produced for centres. These reports are intended to enable teachers to better prepare their students for future examinations (OCR, 2012). The precise content of these reports varies depending on the subject and awarding organisation, but all generally contain a commentary on the way that students answered the questions. In Mathematics this commentary may refer to the methods that students used to answer the question, but it often also includes details about the common errors that were made by students.

There is a debate in the literature about the difference between errors and misconceptions. Confrey (1990) defines both errors and misconceptions as resulting from the rules and beliefs that students hold, but suggests that the difference is that misconceptions are attached to particular theoretical positions. However, other researchers, such as Nesher (1987), use the term misconceptions to describe systematic errors without reference to a theoretical position. Further researchers, such as VanLehn (1982, in Confrey, 1990) and Brown and Burton (1978, in Dickson *et al*., 1984), use a further term, 'bug', to describe those errors that arise from wrong steps in a calculation procedure.

There is a body of research literature that identifies misconceptions in students' mathematical understandings. For example, Swan (1990) described two sets of misconceptions held by students: those that affected their calculations using the four operations (addition, subtraction, multiplication and division); and those affecting their interpretation of graphs. Other researchers have investigated misconceptions that occur in algebra (e.g. Brown & Burton, 1978). However, many of these studies were carried out in the 1970s and 1980s, and the misconceptions that were identified then may not be as relevant today. Changes to the content of specifications, or the way that Mathematics is taught may have affected the errors that students make, and there may now be previously unidentified misconceptions as a result.

The aim of this study was to identify common errors that have been made in Mathematics exams. It examined all the common errors that students made, regardless of whether they were systematic errors, bugs or slips, as all types could provide useful information for teachers and examiners. It focussed on the General Certificate of Secondary Education

(GCSE) and the International General Certificate of Secondary Education (IGCSE) so that the results could feed into the redevelopment of GCSE Mathematics.

Methodology

Three Mathematics specifications were used in this study: IGCSE Mathematics (0580), GCSE Mathematics A (J512) and GCSE Mathematics B (J567). It was necessary to select two OCR Mathematics GCSE qualifications because a change in specification for first teaching in 2010 meant that there was not a single qualification that covered the entire period. These two qualifications were selected because J512 was a linear specification, and J567 was the new version of that specification. The linear nature of the specifications was important as it limited the number of papers that needed to be investigated; the modular GCSE specification would have had too many papers to investigate in the time available.

Copies of the examiners' reports and exam papers were obtained for all three qualifications. It had initially been hoped that it would be possible to investigate both the June and the November sessions, but this was not possible due to the number of papers that would have had to be investigated. Instead the June sessions were chosen, as this was the main session for candidates taking the exams in England. Cambridge International Examinations' (CIE) IGCSE Mathematics qualification is offered around the world, and different versions of the papers are produced for different time zones. For this particular study, the examiners' reports and question papers for the time zone that included England were investigated[1], as this enabled a direct comparison between the GCSE and IGCSE qualifications. Details of the papers examined are contained in Appendix 1.

Within each examiner's report, any common errors that candidates made were identified. It should be noted that the examiners may have used the term "common error" in different ways. In some instances it may have referred to errors that were made by a high proportion of the candidates. Alternatively, it may have described a high proportion of the incorrect answers for that question. In this case, if most candidates got the question right then the common error may only have been made by a small proportion of the candidates. Whilst the examiners would know which of these applied, as they had access to the item statistics for all the questions, the reports do not always make explicit the proportion of candidates that were affected.

Each error was coded against a theme, sub-theme and example. For the purposes of this study, comments were coded as common errors either when the examiner had referred to them as such, or when the examiner had commented on errors that were made by a number of candidates. The themes were taken from a list of subject areas in Mathematics that were developed by Mathematics specialists within OCR. Additional themes were added during the analysis where it was not possible to find an appropriate theme. More detailed sub-themes emerged from the data and were used to enable more detailed coding of particular issues. The themes and sub-themes do not necessarily correspond to the grouping of topics in the specifications. The question number, paper and year for each common error were also recorded. The coding was initially carried out separately for each paper, to make it easier to record the detail that was necessary.

Once the coding for the individual papers had been completed, the coding was combined to allow any common errors affecting multiple papers, years and qualifications to be identified. Errors that affected more than one question were identified, and these are summarised in this article. The common errors have been listed alphabetically by theme (in bold) and sub-theme (in italic). The papers that they affected are identified in the summaries so that it is clear whether they occurred on GCSE or IGCSE papers, and whether they affected foundation/core tier candidates or higher/extension tier candidates[2]. For the GCSE papers, it has been noted whether the errors occurred on calculator or non-calculator papers when this was considered important for the way in which candidates answered the questions. This was not coded for the IGCSE papers, as they are all calculator papers.

Algebraic fluency

Manipulating expressions

Candidates made several common errors when manipulating expressions and equations. When expanding brackets, a common error made by both foundation tier GCSE candidates and core tier IGCSE candidates was to only multiply out part of the brackets (e.g. expanding $3(2x-5)$ to give $6x-5$). Another common error made by core tier IGCSE candidates was to forget to change the sign when multiplying out brackets, particularly for the second term in the brackets (e.g. multiplying out the second term in $-4(x-y)$ as $-4y$).

Core tier IGCSE candidates found it difficult to simplify expressions involving indices. Common errors were to use the wrong operation for the integers and/or indices, and not to realise that a term without a power needed to be considered as an index of 1 when simplifying the indices.

Question	Examiners' Report comment
15 Simplify **(a)** $3p \times 5p^3$ **(b)** $24q^2 \div 8q^{-3}$ [CIE, June 2009, Paper 11]	*"Adding instead of multiplying the integers and not adding 1 to 3 for the index were common errors. Similar errors often spoilt part (b), namely subtracting integers an adding, or even attempting to divide, the indices."*

Common errors made by the higher tier GCSE candidates were to multiply indices that should have been added, and to divide indices that should have been subtracted. Some higher tier GCSE candidates also carried out the subtraction of the indices in the wrong order, so that they obtained a positive power rather than a negative one. When the powers occurred both inside and outside a bracket (e.g. $(3x^4y)^2$) candidates found simplifying the expression more difficult, often adding the indices rather than multiplying them together. .

The higher tier GCSE candidates found indices particularly problematic when they occurred as part of a fraction (e.g. $\frac{2^{3x+2}}{2^{x+5}}$). Most candidates did not know what to do, and whilst the better candidates realised that they had to subtract the powers, most did so incorrectly. Common errors were

1. CIE papers have been offered in time zones since 2009. In 2009 two versions of Papers 1 and 2 were produced, but Papers 3 and 4 were not split into time zones. England fell into the first time zone, so English students would have sat papers 11, 21, 3 and 4. From 2010 onwards, three different time zones were used, and all papers had time zone variants. Students in England would have sat papers 12, 22, 32 and 42.

2. GCSE Mathematics papers are split into foundation and higher tier; IGCSE Mathematics papers are split into core and extension tier.

to add the numerical terms together, or to find the correct numerical term but not include it in the power (i.e. obtaining answers of 2^{2x+7} or $2^{2x}-3$ from $\frac{2^{3x+2}}{2^{x+5}}$)

Extension tier IGCSE candidates also had problems simplifying expressions that involved fractions and indices. A common error was to subtract numerical terms/values rather than divide by them, and/or to incorrectly simplify powers (e.g. simplifying $\frac{8x^6y^5}{2x^4y}$ as $6x^4y^4$ or $4x^2y^5$). The extension tier IGCSE candidates found simplifying expressions containing fractions and negative cubed roots (e.g. $\left(\frac{27}{x6}\right)^{-\frac{1}{3}}$) particularly problematic. Many candidates could find the cube root of individual numeric and algebraic terms in the equation (e.g. finding the cube root of 27 and/or x^6), but did not know what to do with the negative sign in front of the cube root.

Factorisation also caused problems for candidates. A common error made by GCSE foundation tier candidates was only partially factorising expressions (e.g. only taking out one factor from an expression when there were two). The higher tier GCSE candidates sometimes added or multiplied terms instead of factorising them even when this is not possible (e.g. trying to add the terms in the expression $2a^2+8ab$). Both the higher tier GCSE candidates and the extension tier IGCSE candidates did not always notice when an expression was a difference of two squares, leading to incorrect factorisations. When they did notice this, they often made errors in finding the numerical values (e.g. dividing the numerical value by two rather than finding the square root, placing one value outside the brackets).

Question	Examiners' Report comment
11(a) Factorise (i) $4x+14$ (iii) x^2-16 [OCR , June 2011, Paper 4]	*"Common errors in part (a)(i) were $4(x+3.5)$ and $2(x+7)$ and in part (a)(iii) $(x+8)(x-8)$, $x(x-16)$ or $(x-4)^2$."*

The extension tier IGCSE candidates sometimes used the quadratic formula to find the roots of equations when factorising, but then made errors in their factorisation.

Question	Examiners' Report comment
6(c) Simplify $\frac{x^2-16}{2x^2+7x-4}$ [CIE, June 2012, Paper 42]	"Many of these gave the factors as $(x-\frac{1}{2})(x+4)$ and lost the 2 from the $2x-1$."

A common error made by IGCSE extension tier candidates in expressions set out as a fraction (e.g. $\frac{x^2-16}{2x^2+7x-4}$) was to attempt to cancel terms without factorising first. GCSE higher tier candidates also made errors in these expressions, often cancelling out the terms in brackets and ignoring the powers that accompanied them (e.g. simplifying $\frac{2(x-1)^2}{(x-1)}$ to 2^2).

Rearranging equations and formulae

Core tier IGCSE candidates commonly got the sign wrong when moving terms from one side of the equation to the other by addition or subtraction (e.g. making a positive term negative or vice versa). They also did not apply division and multiplication to the whole equation correctly (e.g. changing $z=2x-y$ into $x=\frac{z-y}{2}$ or $x=2z-y$).

Higher tier GCSE candidates sometimes confused operations when rearranging equations, (e.g. subtracting when they should have divided). Another common error was for them to rearrange equations in the wrong order (e.g. to take a square root of one term only before that term was isolated), or to remove terms from inside a bracket before the bracket had been expanded.

Question	Examiners' Report comment
13(d) Rearrange this formula to make r the subject $S=4\pi r^2$ [OCR, June 2012, Paper 4]	*"The square root was dealt with correctly but often before the 4π. Many subtracted the 4π instead of dividing and those who did divide often did two separate divisions."*
13(b) Rearrange this formula to make h the subject $A=2\pi r(r+h)$ [OCR, June 2010, Paper 3]	*"The majority incorrectly tried to move r or h from within the bracket before either expanding or dividing."*

Writing expressions and equations from descriptions

All candidates found it difficult to write down expressions and equations, either from worded descriptions, or from diagrams of shapes. There were many misconceptions about the way that numeric and algebraic terms were combined in equations/expressions. Some weaker candidates did not know that $2y$ is $2\times y$. Other candidates thought that $y+y=y^2$ or $y\times 6y=7y$. Some candidates combined terms that should not be combined (e.g. writing $w+z$ as wz or $15a+32$ as $47a$). Another common error was to confuse an expression with an equation. This could prove problematic if they had written an expression rather than an equation and a later part of the question required them to use the equation that they had written (e.g. rearranging it or solving it).

Substituting into expressions and solving equations

One of the most common issues with solving equations was candidates using trial and error to find the answer, rather than solving the equation algebraically. Some of these candidates obtained correct answers, but mistakes were common.

There were two common errors made by foundation tier GCSE candidates when substituting values into expressions. They sometimes used the wrong operation, which led to them adding values they should have subtracted. Some candidates also seemed confused by the process of substitution, adding together the values of the numerical and algebraic terms rather than multiplying.

Question	Examiners' Report comment
13(a) Work out the value of $3a-4b$ when $a=5.5$ and $b=2$ [OCR, June 2012, Paper 2]	*"Lower scoring candidates tended to add the 3 and 5.5 and also 2 and 4, giving an answer of 2.5."*

IGCSE core tier candidates also made errors when substituting into expressions. Their common errors included not working out the part of the equation in brackets first, and adding a value that occurred after the bracket when it should have been used to multiply. Other common errors made by IGCSE core tier candidates were to ignore negative signs in the numbers that they substituted in, and to carry out calculations in the wrong order (i.e. ignoring rules about the order of operations).

In substituting numbers into expressions, higher tier GCSE candidates found it difficult to substitute negative numbers and fractions into equations containing powers. Common errors included giving negative values for the square of negative numbers (e.g. writing $(-2)^2=-4$), and not knowing how to square fractions (e.g. thinking that $\left(\frac{1}{2}\right)^2=1$ or 2.5).

Solving equations

Higher tier GCSE candidates made errors when there were fractions in the equations that they were solving (e.g $\frac{x}{2} - 3 = 5$). A common error was to divide the whole equation by the denominator rather than to multiply by it. Another error was to only multiply part of the equation by the denominator (e.g. multiplying by the denominator in $\frac{x}{2} - 3 = 5$ to give $x - 3 = 10$).

GCSE higher tier candidates taking the non-calculator paper commonly made errors in their numerical calculations when solving equations. Some candidates confused multiplication and division; others confused positive and negative answers. Some candidates appeared not to know what to do when the result of a division was not an integer (e.g. $2x - 3 + 0$) and commonly gave the numerical value from the equation as their answer instead (e.g. solving $2x - 3 = 0$ to give $x = 3$).

When solving simultaneous equations, the core and extension tier IGCSE candidates and higher tier GCSE candidates commonly failed to consistently add (or subtract) all the terms in the equations. When equations did not appear in the standard format, some extension tier IGCSE candidates unnecessarily rearranged them and made errors in the rearrangement or subsequent solution. Higher tier GCSE candidates also commonly lost marks through poor arithmetic or by choosing the wrong method to solve the simultaneous equations (e.g. adding equations that should have been subtracted or equated).

The higher tier GCSE and extension tier IGCSE papers also required candidates to solve quadratic equations. The higher tier GCSE candidates appeared not to know the methods for solving quadratic equations (particularly the quadratic formula), or could not use them to find the correct answers. The IGCSE extension tier candidates appeared to know the quadratic formula, but when using it either lost marks for accuracy or made mistakes with negative signs.

The higher tier GCSE and extension tier IGCSE candidates also made common errors on the questions with inequalities in them. A common error amongst both sets of candidates was to change the inequality signs either into a different (incorrect) inequality sign, or into an equals sign. Candidates on the extension tier IGCSE did not recognise that an instruction to "solve $9 < 3n + 6 \leq 21$ for integer values of n" meant that they needed to list the values, and therefore they lost marks for leaving the answer as an inequality.

Appropriate use of calculator

A common error made by core tier IGCSE candidates was to incorrectly input calculations into their calculators, which meant that the order of operations was incorrect.

Question	Examiners' Report comment
1 Work out the value of $\frac{48}{19.1 - 3.5 \times 4.6}$ [CIE, June 2012, Paper 12]	*"A common error was 48 ÷ 19.1 – (3.5 × 4.6) to give –13.5. Also seen a number of times was the denominator worked as (19.1 – 3.5) × 4.6."*

Candidates taking the IGCSE extension paper made errors when squaring negative numbers on their calculators. They commonly input the calculation without brackets (i.e. calculating -5^2 instead of $(-5)^2$).

Coordinate geometry and transformations

Coordinate geometry

Candidates on the higher tier GCSE paper made errors with 3D coordinates. A common error was to confuse the order of the coordinates when writing them down. Some candidates wrote the value for the z axis second, possibly confusing it with the vertical y axis in 2D coordinates.

Transformations

Both GCSE and IGCSE candidates on all papers struggled with describing transformations. One of the most common errors was to misread the instructions in the questions and to give combinations of transformations rather than the single transformation that was required by the question.

For line symmetry, candidates on all papers reflected shapes in the wrong line when they were required to draw a reflection, or drew a translation instead. When identifying reflections they found it difficult to give the equation of the line that the shape was reflected in. Common incorrect lines for the reflection were the x axis or the y axis. Similarly, when asked to describe rotational transformations, candidates often gave an incorrect centre of rotation.

The enlargement questions also caused problems for candidates. When describing enlargements, core tier IGCSE candidates often omitted the centre of enlargement from their descriptions. They also found fractional and negative enlargements problematic, confusing the two.

Question	Examiners' Report comment
5(e) Draw the enlargement of shape A, centre (-4, 8), with scale factor $\frac{1}{2}$ [CIE, 2009, Paper 3]	*"Whilst most candidates were able to demonstrate their understanding of enlargement, often this was with scale factors of either 2 or ½ or from a different centre. A significant number omitted this part of the question."*

Extension tier IGCSE candidates were able to draw and describe enlargements. However, they could not calculate enlargements of area or volume correctly, when given the enlargement of a line. They used the linear scale factor, rather than squaring (for area) or cubing (for volume).

The extension tier IGCSE candidates found transformations involving stretch and sheer difficult, and often confused the two transformations. Another error was to identify stretch or sheer as an enlargement instead. Candidates did not always describe the invariant line correctly.

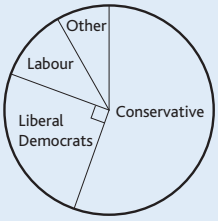
Question	Examiners' Report comment
4(b) Describe fully the single transformation that maps **(ii)** triangle *T* onto triangle *V*. [CIE, June 2010, Paper 42]	*"...an incomplete invariant line e.g. invariant in x rather than the x-axis. Answers such as the x-axis or parallel to the x-axis were not accepted as invariance was not clear."*

Data handling, statistics, probability and chance

Drawing charts and graphs

Candidates on all papers of the GCSE and IGCSE qualifications are expected to draw and interpret charts as part of their data handling skills. The errors that they made depended on the level of the paper, and whether they were GCSE or IGCSE candidates.

Foundation tier GCSE candidates found it difficult to use pie charts to answer questions. Many found it difficult to find the numbers represented in a sector of the pie chart.

Question	Examiners' Report comment
17(c) In one constituency 53 520 people voted in the 2010 general election. This pie chart summarises the results.  **(iii)** How many people voted Conservative? [OCR, June 2012, Paper 2]	*"A common wrong answer was 26 760 from just halving the 53 520. Some did gain credit for the angle of 200°. A common error was to calculate for an angle of 202.5° from 180° + 22.5° rather than using their protractor to measure the angle."*

Higher tier GCSE candidates commonly drew histograms incorrectly. Whilst the width of the bars was usually correct, the heights were usually incorrect because the frequency density was calculated incorrectly. The common error was to treat all classes as though they were of equal width when calculating frequency density (dividing all the frequencies by the same number).

IGCSE candidates and higher tier GCSE candidates made errors when drawing lines of best fit onto scatter graphs. The common errors were joining the points instead of drawing a line of best fit, or drawing a line of best fit that went through the origin. The IGCSE core tier candidates had difficulty describing the correlations that were shown on scatter graphs. Candidates either gave lengthy word descriptions instead of the proper terms (earning them no marks), or gave more detail than was necessary.

Mean, median, mode and range

GCSE and IGCSE candidates at all levels made errors when working with these statistics. Core tier IGCSE candidates commonly confused mean, median and mode. The GCSE foundation tier candidates and IGCSE core tier candidates also had a poor understanding of range, commonly writing down the smallest and largest value, rather than calculating the difference between the two.

All candidates had problems finding the median of data when there was an even number of values. Candidates often gave one of the two middle values as their answer, instead of calculating the mean of the middle two numbers.

Candidates on all papers also found it difficult to find the mean of grouped data. Two common errors were observed in these questions. Candidates often divided incorrectly to calculate the mean, usually either dividing the frequency by the number of classes or using cumulative frequencies. Where the classes represented a range of data, candidates used incorrect mid-points (e.g. the upper bound) to calculate the mean.

Probabilities

Candidates made two types of common error when calculating probabilities. Higher tier GCSE and extension tier IGCSE candidates often appeared not to know whether the sampling was with or without replacement, which led to them using an incorrect denominator in calculating their probabilities. There were further errors made by candidates on all the papers when they calculated combined probabilities. Some candidates used the number of options out of the total possible to give the combined probability, rather than adding/multiplying the probability of each option happening. Other candidates failed to account for all the possible ways in which combined events could occur, leading to incorrect probabilities being calculated. Finally, some candidates got confused about whether the probabilities should be added or multiplied, and used the wrong calculation to find the final probability.

Question	Examiners' Report comment
10(b) Amir tests a laptop at random. Find the probability that both the hard drive and the screen are **not** faulty. [OCR, June 2011, Paper 3]	*"Most candidates identified the correct probabilities part (b) to use in their calculation and those that multiplied them usually reached the correct answer ... The main error was to add the probabilities..."*

Ratio

GCSE candidates on both the higher tier and foundation tier papers found it difficult to carry out calculations involving ratios. A common error made by foundation tier candidates was to multiply (instead of divide) the amount by the ratio that they need to find. The higher tier GCSE candidates divided the amount by the required part of the ratio, rather than by the total of the ratio.

Question	Examiners' Report comment
5(a) In a carton of *Squashy*, orange juice and water are mixed in the ratio of 3:7. How many litres of orange juice are needed to make 60 litres of *Squashy*? [OCR, June 2009, Paper 3; also OCR, June 2009, Paper 1 question 17]	*"Weaker candidates tried to divide 60 by 3 and by 7 7 in part (a)."* (Paper 3) *"The most common working was 60 × 3 = 180, 60 × 7 = 420 giving an answer of 180."* (Paper 1)

Finding correct numerical answers without a calculator

General comments on the GCSE non-calculator papers suggested that candidates' arithmetic skills were poor. The foundation tier candidates had particular problems with subtraction, commonly taking the smaller digit away from the larger one. They also had problems with the order of operations, carrying out calculations in order from left to right, ignoring rules about the order of operations.

Both higher and foundation tier GCSE candidates had problems when decimals were involved in calculations. Candidates on the higher tier could not multiply by decimals correctly, sometimes dividing instead, or on other occasions putting the decimal point in the wrong place.

Fractions, percentages and decimals

Equivalence of fractions, percentages and decimals

Candidates on all the GCSE papers made errors when converting between fractions, decimals and percentages. Candidates often confused tenths and hundredths when converting fractions to decimals (e.g. changing $\frac{3}{50}$

to $\frac{6}{100}$ then writing it as 0.6 rather than 0.06). Higher tier candidates struggled with recurring decimals, often writing degree signs instead of the recurring decimal sign or writing too few digits in the recurring decimal.

Fractions

Candidates on the GCSE non-calculator papers appeared to have several problems when carrying out calculations with fractions without using a calculator. The GCSE foundation tier candidates often failed to put their fractions into simplest form, losing a mark for accuracy. They also did not have an awareness of the relative size of fractions and did not know which numbers to divide and multiply by to find a fraction of a number.

Question	Examiners' Report comment
2(c) Write down a fraction that is smaller than $\frac{1}{10}$ [OCR, June 2009, Paper 1]	*"Roughly half of the candidates not showing an awareness of relative size of a fraction. 0.5/10 was a common answer from weaker candidates."*
11 Work out **(d)** $\frac{5}{6}$ of 78 [OCR, June 2009, Paper 1]	*"Few candidates seemed sure of the method required to do the calculation in part (d). Many were unsure which number, 5 or 6, to divide by (some tried both). Wrong ideas, including 78 – 5 and 78 – 6 were seen"*

The higher tier GCSE candidates found addition and multiplication of fractions without a calculator problematic. Most of the examiners' comments did not identify the exact issues that candidates had, only noting that they were unable to carry out the calculations accurately. However, it was clear that pupils did not have a clear understanding of how to add, subtract, multiply and divide using fractions.

Question	Examiners' Report comment
1(b) $3\frac{1}{3} - 1\frac{5}{6}$ Give your answer in its simplest form [OCR, June 2012, Paper 3]	*"Common errors were doubling $3\frac{1}{3}$ to get $6\frac{2}{6}$, inverting the second fraction and subtracting denominators as well as numerators"*

Percentages

Generally, candidates were able to calculate percentages of numbers, although the GCSE foundation tier candidates were not able to use efficient methods to find them using a calculator.

Question	Examiners' Report comment
12 800 people were asked where they went on their summer holidays. These are the results. UK: 220 | Europe: 450 | Rest of world: 143 | Did not go on holiday: 67 **(a)** Work out the percentage of the people in the survey who replied **(ii)** Rest of the world [OCR, June 2011, Paper 2]	*"Solutions involving long and inefficient methods were often seen in part (a)(ii) and correct answers were rarely seen… Generally candidates did not demonstrate techniques that can be carried out with a calculator in a concise and efficient manner."*

GCSE higher tier candidates and IGCSE core tier candidates found calculating a percentage increase problematic. These candidates were able to find the increased amount, but a common error was to fail to add it onto the total. IGCSE candidates on the extension papers made errors when calculating reverse percentages. Commonly they calculated the percentage of the value they were given instead of using the percentage to find the original value.

Question	Examiners' Report comment
1(b) **(ii)** The money remaining from the \$150 is $37\frac{1}{2}$ % of the cost of a day trip to Cairo. Calculate the cost of the trip [CIE, June 2009, Paper 4]	*"However many candidates calculated $37\frac{1}{2}$ % of \$30." [\$30 was the money remaining].*

Candidates taking the IGCSE core and extension papers had to solve problems involving simple and compound interest. The common error in these questions was to get the two types of interest confused.

Functions and graphs (linear, quadratic, cubic, trigonometric, exponential)

Functions and graphs

Some of the foundation tier GCSE and core tier IGCSE candidates lost marks when drawing graphs of functions. Common errors occurred when drawing the lines between points. Some candidates omitted lines altogether or drew them freehand. Others joined the dots with straight lines instead of drawing a smooth curve, or drew bumpy curves.

Some core tier IGCSE candidates did not know what the equations for curved and straight line graphs looked like, commonly not recognising the link between the equation that they were given and the graph. When asked to write down the equation of a straight line, candidates could identify where it crossed the axis, but could not give the correct equation. They used the wrong axis in their equations (e.g. $y = -0.5$ instead of $x = -0.5$), omitted the negative sign (e.g. $x = 0.5$ instead of $x = -0.5$), made y a multiple of x (e.g. $y = 0.5x$ instead of $x = -0.5$) or wrote an expression rather than an equation (e.g. $x - 0.5$ instead of $x = -0.5$). Higher tier GCSE candidates and extension tier IGCSE candidates made mistakes when calculating gradients of lines. They also commonly wrote fractional gradients as negative when they should have been positive (e.g. $-½x$ instead of $½x$).

Higher tier GCSE candidates and IGCSE candidates also had difficulty with solving equations graphically. The common error made by candidates on both papers was to solve the equations algebraically instead.

Speed

Candidates on the foundation tier GCSE paper found questions about speed difficult. Some candidates who knew the formula for speed and wrote down a distance speed time triangle then used the wrong operation (e.g. multiplication rather than division) to find the answer. Other candidates did not know the formula, so could not answer the question. The extension tier IGCSE candidates did not know how to find distance from a linear, non-horizontal speed-time graph. They incorrectly used the formula (taking values from the graph) rather than finding the area under the graph.

Geometry (including trigonometry and Pythagoras)

Names and properties of shapes, lines and angles

Foundation tier GCSE candidates and core tier IGCSE candidates sometimes made errors in recognising shapes. Commonly confused shapes included trapezium and parallelogram, hexagon and octagon, scalene and isosceles triangles, and isosceles and equilateral triangles. The core tier IGCSE candidates also found it difficult to recognise the labelling of angles, commonly identifying incorrect angles when three letter angle notation was used.

Foundation tier GCSE candidates and core tier IGCSE candidates confused area and perimeter or area and volume. These candidates also made errors when calculating the volumes of prisms. The IGCSE candidates commonly used an incorrect formula in their calculations. Some GCSE candidates were not able to use the formula they had been given to work out the volume, possibly because they did not know how to calculate the cross-sectional area.

When working with circles or cylinders, a common error on all GCSE papers and the core tier IGCSE papers was to confuse diameter and radius. The formulae for circumference and radius of a circle were commonly confused by candidates on all papers.

The properties of circles proved problematic for the core tier IGCSE candidates in questions where they had to find/calculate indicated angles. Candidates often made incorrect assumptions about the lines or shapes that were shown in the diagram (e.g. failing to recognise a line as the diameter or assuming that a triangle was isosceles). Extension tier IGCSE candidates often knew the properties of lines and circles and were able to use them to find angles. Their common error was not using the correct terminology to describe their reasoning (e.g. using terms like z angles instead of alternate angles).

Core tier IGCSE and foundation tier GCSE candidates did not always know the sum of angles in shapes and on straight lines. GCSE foundation tier candidates did not know the angle sum of the interior angles in triangles and quadrilaterals, confusing 180° and 360°. IGCSE core tier candidates commonly confused exterior and interior angles. They also used 180° instead of 360° in the formula for calculating the size of exterior angles. Some GCSE candidates thought all the angles on a horizontal line added up to 180°, regardless of whether the angles were around the same point, or related by properties of lines (e.g. whether they were corresponding or alternate angles).

Loci

Questions about loci proved problematic for candidates on all papers. Candidates did not know which construction to use, and commonly drew incorrect ones to solve problems. Finding angle bisectors appeared particularly problematic. In addition, some candidates lost marks because they did not use compasses.

Trigonometry

GCSE higher tier candidates and IGCSE candidates on both tiers sometimes did not know which trigonometric ratio to use to answer questions. Candidates assumed that triangles were right angled when they were not, or failed to recognise right angled triangles in more complex shapes. Another error made by GCSE higher tier candidates was to calculate the cosine of an angle when they needed to find the inverse cosine.

Bearings

Candidates on all papers made both possible common errors when measuring bearings. They failed to measure bearings from north and they measured the bearings in an anticlockwise direction instead of clockwise.

Knowledge of realistic answers

Foundation/core tier candidates at both GCSE and IGCSE often failed to consider whether the answer that they obtained was realistic.

Matrices

Matrices only appeared on the IGCSE extension tier papers, but they led to several common errors. Candidates knew how to find an inverse of a matrix, but made errors rearranging the matrix. Several of the matrix questions involved algebraic expressions, and in these questions candidates sometimes failed to simplify their expressions. Often this occurred because the candidate had given their answer as a 2×2 matrix, when a 2×1 matrix was the correct answer.

Measures

Candidates across all the papers gave incorrect units in their answers. Foundation tier GCSE candidates commonly gave either cm or cm^2 instead of cm^3 as the unit for a volume they had calculated. Another common error was for candidates to misread the type of units that were used in the question and to give the wrong unit in their answer (e.g. giving an answer in cm^2 instead of $feet^2$).

Conversion between units also caused problems across all the papers. IGCSE and GCSE candidates on both tiers commonly forgot to convert their answers to different units after completing their calculations. The GCSE foundation tier candidates made mistakes when converting distances between cm and km, often giving answers in the wrong order of magnitude.

Question	Examiners' Report comment
15(b) Parvinder has a bicycle. Each wheel has a diameter of 65.5cm. On one journey each wheel rotated 3509 times. Calculate the distance Parvinder cycled. Give your answer in kilometres. [OCR, June 2012, Paper 2]	*"Some attempted to convert the units but most were divided by just 1000 or 10 000 rather than 100 000."*

Higher tier GCSE and extension tier IGCSE candidates struggled to convert units when calculating areas or volumes. The common error was using the linear conversion (e.g. 10cm = 0.1m) rather than squaring or cubing these conversions.

All candidates appeared to have problems with questions about the bounds of measurements. Identifying the upper bound was problematic for GCSE higher tier candidates and IGCSE core tier candidates. They often gave incorrect answers such as 12.4 or 12.49 instead of the correct answer of 12.5. (Generally candidates were able to identify the lower bound correctly.) The IGCSE candidates made errors when they used bounds to find the maximum and/or minimum values for calculations.

Common errors were when candidates calculating the bounds after completing the calculation (i.e. giving the maximum and minimum values for the final answer only), or choosing the wrong bounds to find the maximum difference.

There appeared to be a number of errors when candidates were asked questions involving times on a calculator paper. A very common error (made by all IGCSE candidates and higher tier GCSE candidates) was to calculate a time assuming that there were 100 minutes in an hour rather than 60. Working with times over two days caused problems for the core tier IGCSE candidates, who often failed to give the correct day of the week. Generally they were out by one day, or gave the number of days rather than the name of the day.

Question	Examiners' Report comment
3 The ferry from Helsinki to Travemunde leaves Helsinki at 17 30 on a Tuesday. The journey takes 28 hours 45 minutes. Work out the day and time that the ferry arrives in Travemunde. [CIE, June 2012, Paper 12]	*"It was common to see Thursday, as well as the number, 1 or next day, rather than the correct day of the week [Wednesday]."*

Powers and roots

Foundation tier GCSE candidates on both calculator and non-calculator papers commonly doubled numbers instead of squaring them, and multiplied numbers by three rather than cubing them. When calculating the roots of numbers, these candidates often divided the numbers by 2 or 4 instead of calculating the square root.

Fractional and negative powers proved problematic for core tier IGCSE candidates and higher tier GCSE candidates. A common error was to divide by 2 when the square root was written as a fraction (e.g. $16\frac{1}{2}$) rather than calculating the square root of the number. Negative powers were commonly interpreted as giving the same number as positive powers but making the answer negative.

Question	Examiners' Report comment
18(a) Evaluate. $64^{\frac{1}{2}} \times 2^{-4}$ [OCR, June 2010, Paper 3]	*"$64^{\frac{1}{2}}$ was often given as 32 and 2^{-4} became -8, -16 or 0.0002."*

Some IGCSE candidates on the extension paper did not know how to calculate using indices when the answer had to be left in the same form. The common error was to think that the rule of multiplying the bases and adding the indices applied when the numbers had different bases.

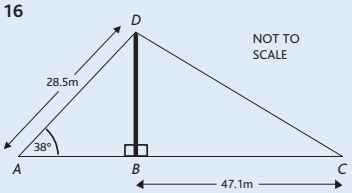
Question	Examiners' Report comment
5 Write $2^8 \times 8^2 \times 4^{-2}$ in the form 2^n. [CIE, June 2010, Paper 22]	*"...about a quarter of the candidates multiplied the bases together and then added the indices."*

GCSE higher tier candidates did not know how to multiply powers over brackets, commonly adding the indices (e.g. writing the answer for $(7^3)^5$ as 7^8).

In general the examiners' comments suggested that GCSE higher tier candidates were not familiar with the rules concerning surds. Common errors included leaving a square root in the answer when it could have been solved to find a whole number (e.g. writing $\sqrt{9}$ instead of 3), and not simplifying surds (e.g. leaving an answer as $\sqrt{18}$ instead of $3\sqrt{2}$).

Rounding

GCSE and IGCSE candidates made several common errors when required to round their answers. The most frequent error, made by GCSE and IGCSE candidates on both tiers, was failing to give calculator answers to an appropriate number of decimal places (or significant figures). A common error made by the IGCSE candidates was to round incorrectly, or to round too early in their calculations.

Question	Examiners' Report comment
16  A flagpole, BD, is attached to level horizontal ground by ropes, AD and CD. AD = 28.5m, BC = 47.1 m and angle DAB = 38°. Calculate **(a)** BD, the height of the flagpole [CIE, June 2011, Paper 12]	*"Unfortunately some lost a mark by presumably rounding 17.546 to 17.55 and then 17.6 to 3 significant figures."*

Finally, questions on the IGCSE extension papers that required candidates to show that answers were equivalent to a rounded value caused problems for candidates. Often candidates failed to show a more accurate value before giving the rounded value, which meant they lost the accuracy mark.

Question	Examiners' Report comment
6 A spherical ball has a radius of 2.4 cm. **(a)** Show that the volume of a ball is 57.9 cm^3, correct to 3 significant figures. [CIE, June 2010, Paper 42]	*"The accuracy mark was often lost as the majority wrote down the answer of 57.9 given in the question. It is important for candidates to understand that if they are required to show how to obtain an answer that is given in the question and is not exact, then a more accurate answer is needed for full marks."*

Sequences and patterns

Candidates at both GCSE and IGCSE found it difficult to give term-to-term rules and the nth term rule for sequences. A common error was to confuse the two. When asked to give the nth term rule, candidates often incorrectly gave the term-to-term rule algebraically instead. Another common error was to give the terms in the nth term rule in the wrong order (e.g. giving 21-4n instead of 4n-21).

Sets

Extension tier IGCSE candidates confused the union and intersect symbols and failed to shade the correct regions of Venn Diagrams.

Standard form

Both GCSE and IGCSE candidates at all tiers found it difficult to write and calculate with numbers in standard form. Answers were commonly written with only the number and power (e.g. 3^8 instead of 3×10^8) or with the decimal point in the wrong place (61.4×10^5). Another common error was to round to the required order of magnitude but to forget to write it in standard form (e.g. 61,400,000 instead of 6.14×10^8).

Terminology

Candidates on all the GCSE papers and candidates taking the core tier IGCSE papers appeared to be confused about factors and multiples, and it was common for candidates to give the wrong one. The highest common factor and lowest common multiple were also sometimes confused, with candidates giving the lowest common factor instead of the lowest common multiple.

Vectors

Core tier IGCSE candidates found vector notation problematic. When using vectors as part of translations, common errors were to misread the scale, to get negative and positive directions confused and to present the translation as coordinates instead of a column vector. Higher tier GCSE candidates using vector notation confused the position and/or the signs of the two values.

The IGCSE extension tier candidates did not recognise some of the properties of vectors. They did not recognise the modulus of vector notation, and they did not realise that a position vector is given from the origin. The comments from examiners suggested that candidates needed encouragement to show their route when answering questions about vectors.

Conclusion

This article has identified the common errors that candidates have made in many different areas of Mathematics. The information may be used to inform future teaching, as the errors can be used to ensure that candidates have a better understanding of the mathematical skills and knowledge that they will need in examinations. It is also useful for awarding organisations as they can use it in the development of future assessment material to ensure that the questions address areas of students' understanding that may not be secure. In addition, support materials and resources can be developed that address the common errors and enable students to acquire a better understanding of Mathematics.

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Examiners' Reports

OCR GCSE (J512 AND J567)

J512 June 2009 http://www.ocr.org.uk/Images/62540-examiner-s-report-june.pdf

J512 June 2010 http://www.ocr.org.uk/Images/63805-examiner-s-report-june.pdf

J512 June 2011 http://www.ocr.org.uk/Images/63537-examiners-reports-june.pdf

J567 June 2012 http://www.ocr.org.uk/Images/59280-examiners-reports-june.pdf

(All available online; accessed 20 November 2013).

CIE IGCSE (0580)

June 2009 http://teachers.cie.org.uk/docs/dynamic/32880.pdf

June 2010 http://teachers.cie.org.uk/docs/dynamic/39775.pdf

June 2011 http://teachers.cie.org.uk/docs/dynamic/47607.pdf

June 2012 http://teachers.cie.org.uk/docs/dynamic/56160.pdf

(All available online; accessed 20 November 2013).

Question Papers

OCR GCSE (J512 AND J567)

J512 June 2009

Paper 1 http://www.ocr.org.uk/Images/66846-question-paper-unit-j512-01-paper-1-foundation-tier.pdf

Paper 2 http://www.ocr.org.uk/Images/65612-question-paper-unit-j512-02-paper-2-foundation-tier.pdf

Paper 3 http://www.ocr.org.uk/Images/60182-question-paper-unit-j512-03-paper-3-higher-tier.pdf

Paper 4 http://www.ocr.org.uk/Images/65613-question-paper-unit-j512-04-paper-4-higher-tier.pdf

J512 June 2010

Paper 1 http://www.ocr.org.uk/Images/61421-question-paper-unit-j512-01-paper-1-foundation-tier.pdf

Paper 2 http://www.ocr.org.uk/Images/65034-question-paper-unit-j512-02-paper-2-foundation-tier.pdf

Paper 3 http://www.ocr.org.uk/Images/59596-question-paper-unit-j512-03-paper-3-higher-tier.pdf

Paper 4 http://www.ocr.org.uk/Images/63231-question-paper-unit-j512-04-paper-4-higher-tier.pdf

J512 June 2011

Paper 1 http://www.ocr.org.uk/Images/65914-question-paper-unit-j512-01-paper-1-foundation-tier.pdf

Paper 2 http://www.ocr.org.uk/Images/58670-question-paper-unit-j512-02-paper-2-foundation-tier.pdf

Paper 3 http://www.ocr.org.uk/Images/62305-question-paper-unit-j512-03-paper-3-higher-tier.pdf

Paper 4 http://www.ocr.org.uk/Images/65915-question-paper-unit-j512-04-paper-4-higher-tier.pdf

J567 June 2012

Paper 1 http://www.ocr.org.uk/Images/134458-question-paper-unit-j567-01-paper-1-foundation-tier.pdf

Paper 2 http://www.ocr.org.uk/Images/134459-question-paper-unit-j567-02-paper-2-foundation-tier.pdf

Paper 3 http://www.ocr.org.uk/Images/134460-question-paper-unit-j567-03-paper-3-higher-tier.pdf

Paper 4 http://www.ocr.org.uk/Images/134461-question-paper-unit-j567-04paper-4-higher-tier.pdf

(All available online; accessed 27 November 2013).

CIE IGCSE (0580)

June 2009

Paper 1 Paper 2 Paper 3 Paper 4

June 2010

Paper 1 Paper 2 Paper 3 Paper 4

June 2011

Paper 1 Paper 2 Paper 3 Paper 4

June 2012

Paper 1 Paper 2 Paper 3 Paper 4

(All available online from Cambridge International Examinations through their secure *CIE TeacherSupport* website).

Appendix 1: Descriptions of the qualifications

OCR Mathematics A GSCE (J512)

OCR Mathematics A GSCE (J512) was offered until the June 2011 session and was comprised of four papers. Students had to take either two papers at the foundation tier (Paper 1 and Paper 2) or two papers at the higher tier (Paper 3 and Paper 4).

Table 1 GCSE Mathematics papers (J512)

Paper	*Description*	*Calculator*	*Duration*	*Weight*
1	Mathematics Paper 1 – Foundation	No	2 hours	50%
2	Mathematics Paper 2 – Foundation	Yes	2 hours	50%
3	Mathematics Paper 3 – Higher	No	2 hours	50%
4	Mathematics Paper 4 – Higher	Yes	2 hours	50%

OCR Mathematics B GSCE (J567)

OCR Mathematics B GSCE (J567) is comprised of four papers. Students must take either two papers at the foundation tier (Paper 1 and Paper 2) or two papers at the higher tier (Paper 3 and Paper 4).

Table 2 GCSE Mathematics papers (J567)

Paper	*Description*	*Calculator*	*Duration*	*Weight*
1	Mathematics Paper 1 – Foundation	No	1 hour 30 mins	50%
2	Mathematics Paper 2 – Foundation	Yes	1 hour 30 mins	50%
3	Mathematics Paper 3 – Higher	No	1 hour 45 mins	50%
4	Mathematics Paper 4 – Higher	Yes	1 hour 45 mins	50%

CIE Cambridge IGCSE Mathematics (0580)

CIE Cambridge IGCSE Mathematics (0580) is comprised of four papers. Students must take either two core papers (Paper 12 and Paper 32) or two extension papers (Paper 22 and Paper 42).

Table 3 IGCSE Mathematics papers (0580)

Paper	*Description*	*Calculator*	*Duration*	*Weight*
12	Short answer questions – Core curriculum	Yes	1 hour	35%
22	Short answer questions – Extended curriculum	Yes	1 hour 30 mins	35%
32	Structured questions – Core curriculum	Yes	2 hours	65%
42	Structured questions – Extended curriculum	Yes	2 hours 30 mins	65%