A Level

Mathematics

Session: 1957 June
Type: Question paper
Code: 19

MATHEMATICS 182

ADVANCED LEVEL

PAPER I

(Two hours and a half)

Answers to not more than **eight** *questions are to be given up. A pass mark can be obtained by good answers to about* **four** *questions, or their equivalent.*

Begin each answer on a fresh sheet of paper and arrange your answers in numerical order.

Mathematical tables and squared paper are provided.

1. (i) Solve for x and y the equations

$$2^x 3^y = 5,$$
$$4^x = 7\,(9^y),$$

giving your answers correct to three significant figures.

(ii) By putting $a = -(b+c)$, or otherwise, prove that $(a+b+c)$ is a factor of $a^3+b^3+c^3-3abc$. Deduce a factor of $8x^3+y^3-z^3+6xyz$.

2. Show that if $a>0$ the function ax^2+bx+c has a least value, and find it.

Find the values of a, b, and c, if the function ax^2+bx+c has the minimum value -5 when $x=-1$ and the value zero when $x=-2$. Find the range (or ranges) of values of x for which the function is greater than 75.

3. (i) Find the values of a and n if the first three terms of the binomial expansion of $(1+ax)^n$ are

$$1-4x+\tfrac{20}{3}x^2.$$

(ii) Find the expansion in ascending powers of x, as far as the term in x^2, of the function

$$\frac{3+x}{4-x^2+\sqrt[3]{(8-2x^2)}}.$$

4. (i) Prove that unless $x=1$ the sum of the series

$$S \equiv 1+x+x^2+\ldots+x^{n-1}$$

is $(1-x^n)/(1-x)$.

(ii) By multiplying the series

$$S' \equiv 1+2x+3x^2+\ldots+nx^{n-1} \quad (x \neq 1)$$

by $(1-x)$, and summing the resulting series, find the sum of the series S'.

(iii) Consider what happens to the sum of each of the given series as $n\to\infty$, if $0<x<1$. (It may be assumed that if $0<x<1$ then $nx^n \to 0$ as $n\to\infty$.)

5. Prove, graphically or otherwise, that the equation

$$x\,(x^2+20)=100$$

has only one root and that it lies between 3 and 4.

By drawing graphs of the two functions x^2+20 and $100/x$ for values of x between 3 and 3·5, find the value of the root, correct to two decimal places.

6. (i) Prove that

$$\tan\theta = \frac{\sin 2\theta}{1+\cos 2\theta},$$

and hence obtain $\tan 15°$ and $\tan 67\tfrac{1}{2}°$ as surds, in their simplest forms.

(ii) Prove that

$$\sin\tfrac{1}{2}A - \cos\tfrac{1}{2}A = \pm\sqrt{(1-\sin A)},$$

determining for what range of angles A between 0 and 4π the positive sign is to be taken.

7. A line AP is drawn through the vertex A of a triangle ABC, so that P and C are on opposite sides of AB, and angle $BAP=\theta$. By projecting the sides of triangle ABC on AP, prove that

$$c\cos\theta = a\cos(B-\theta)+b\cos(A+\theta).$$

Deduce from this result the two formulae

$$c=a\cos B+b\cos A, \quad a\sin B = b\sin A.$$

Further, by considering the special case $\theta=A=B$, deduce that $\cos 2A = 2\cos^2 A - 1$.

8. (i) Find the solutions between 0° and 360° inclusive of the equations

(a) $\cot \frac{3x}{2} = \tan 171°$;

(b) $\sin 3x + 3 \cos 2x + \sin x = 3$.

(ii) Find the solutions in radians between 0 and 2π of the equation $\cos 3x = \sin x$.

9. In any triangle ABC, prove the formula

$$\tan \tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

[The cosine formula may be assumed if required.]

Find the angles of triangle ABC, if $a = 11{\cdot}42$ cm., $b = 13{\cdot}71$ cm., and $c = 14{\cdot}07$ cm.

10. From a point A of a straight level road which runs due north, a peak bears due west. A man walks along the road, starting from A and going at a steady speed. After 10 min. the elevation of the peak is α; after a further 20 min. it is β. If the elevation of the peak from A is θ, prove that

$$8 \cot^2 \theta = 9 \cot^2 \alpha - \cot^2 \beta.$$

If the man's speed is 4 m.p.h. and α, β are 17° 18′ and 10° 24′ respectively, calculate the height of the peak above the level of the road, to the nearest 50 ft.

MATHEMATICS 183

ADVANCED LEVEL

PAPER II

(Two hours and a half)

Answers to not more than **eight** *questions are to be given up. A pass mark may be obtained by good answers to about* **four** *questions, or their equivalent.*

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Mathematical tables and squared paper are provided.

1. Prove that the points $A(-2, -1)$, $B(4, 3)$, $C(6, 0)$, $D(0, -4)$ form the vertices of a rectangle.

The straight line $x = 3$ meets the sides AB and DC in P and Q respectively. Calculate the area of the trapezium $PBCQ$.

2. Show that the circle $x^2 + y^2 - 2\lambda x + (\lambda - 2a)^2 = 0$ touches the parabola $y^2 = 4ax$ $(a > 0)$ at two real points, for all values of λ greater than $2a$.

Find the value of λ $(> 2a)$ for which the circle will pass through the focus of the parabola. In this case, if A is the centre of the circle and P is one of the points of contact of the circle and the parabola, show that a second circle having centre P and radius PA will touch the directrix of the parabola.

3. Find the equation of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $(a \cos \theta, b \sin \theta)$ and the equation of the normal to this ellipse at the point $(a \cos \theta, -b \sin \theta)$.

Show that there is a value of θ between 0 and $\pi/2$ for which these two lines intersect on the axis of y, provided that a^2 is greater than $2b^2$.

4. Find the coordinates of the point of intersection of the tangents drawn to the rectangular hyperbola $xy = c^2$ at the points $(cp, c/p)$ and $(cq, c/q)$.

A variable chord of the hyperbola is such that its mid-point lies on a fixed straight line parallel to the axis of y. Show that the point of intersection of the two tangents to the hyperbola at the extremities of the chord lies on a fixed straight line parallel to the axis of x.

5. (i) Differentiate $(1 - 2x)^p e^{mx}$ with respect to x.

(ii) If $y = \log_e \cot\left(\frac{\pi}{4} - \frac{x}{2}\right)$, prove that $\frac{dy}{dx} = \sec x$.

(iii) A point P is moving along the circumference of a circle of centre O and radius r so that the radius OP has constant angular velocity ω. Initially the point is at A and after time

$t\ (<\pi/\omega)$ the point is at B. If the length of the chord AB is x and the area of the triangle AOB is y, show that

$$\left(\frac{dx}{dt}\right)^2 - \omega\frac{dy}{dt} = \tfrac{1}{2}r^2\omega^2.$$

6. Sketch the curve represented by the equation

$$y = \frac{x+1}{x-1}.$$

Show on the same axes the curve represented by the equation

$$y^2 = \frac{x+1}{x-1}.$$

7. The length of the side of a square $ABCD$ is a, and E is a variable point on the side AB, lying between A and B. A particle starts from D and moves with uniform velocity u along the line DE and then moves with uniform velocity v along the line EB. If the angle ADE is θ, find an expression for the total time taken to get from D to B. Show that if $(v/u) > \sqrt{2}$, there is a true minimum value of the time for a certain value of θ.

Calculate the minimum value of the time in terms of a and u, if $3v = 5u$.

8. (i) Integrate with respect to x

(*a*) $\dfrac{2-5\cos^3 x}{3\cos^2 x}$, (*b*) $x\sqrt{9+x^2}$.

(ii) Prove that $\displaystyle\int_3^4 \frac{x^2+4}{x^2-4}\,dx = 1 + 2\log_e\left(\tfrac{5}{3}\right)$.

9. The origin O is joined to the point $P\ (h, k)$ on the parabola $y^2 = 4ax$, and N is the foot of the perpendicular from P to the axis of x. Prove that the area enclosed between the straight line OP and the parabola is one-third of the area of the triangle OPN.

Show also that the x-coordinate of the centre of gravity of the area between OP and the parabola is $2h/5$.

10. Write down the first three non-zero terms in the expansions of both $\sin x$ and $\cos x$ in ascending powers of x.

Hence, or otherwise, prove that the first two non-zero terms in the expansion of $\tan x$ in ascending powers of x are $x + x^3/3$.

In a circle of centre O and radius r, OP and OQ are two radii enclosing a small angle. Tangents to the circle at P and Q intersect at R. If the length of the arc PQ is s, show that the area enclosed by PR, QR and the arc PQ is approximately $s^3/24r$.

Hence show that, if a regular polygon having a large number of sides circumscribes the circle, the difference between the area of the polygon and the area of the circle is approximately $\pi s^2/12$, where s is the length of the arc between any two consecutive points of contact of the polygon and circle.

MATHEMATICS 184

ADVANCED LEVEL

PAPER III

(*Two hours and a half*)

Answers to not more than **eight** *questions are to be given up. A pass mark can be obtained by good answers to about* **four** *questions, or their equivalent.*

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Mechanics

1. A light inelastic string passes over a smooth light pulley and masses of 3 lb. and 5 lb. are attached to its ends. The masses are released from rest with the 5 lb. mass 4 ft. above an inelastic horizontal table. Find the time which elapses from the instant when the masses are first released until the 5 lb. mass hits the table for the second time, assuming that the 3 lb. mass hits neither the table nor the pulley.

2. When a motor-cyclist is travelling along a straight stretch of road from south to north at a steady speed of 30 m.p.h. the wind appears to him to come from a direction N. 40° E. When he returns along the same road at the same speed the wind appears to come from a direction S. 30° E. Find, by drawing or otherwise, the true magnitude and direction of the wind.

3. A projectile is fired with velocity V from the edge of a vertical cliff of height b and hits the sea at a distance a from the foot of the cliff. Show that there are in general two possible directions of projection and that these will be at right angles if

$$bV^2 = ga^2.$$

If $a = 216$ ft., $b = 162$ ft., and $V = 96$ ft. per sec., verify that this condition is satisfied, and find the two possible directions of projection.

4. A car weighing 1 ton can ascend a gradient of $\sin^{-1}\frac{1}{6}$ at a steady speed of 18 m.p.h. Working at the same horse-power it can ascend a gradient of $\sin^{-1}\frac{1}{12}$ at a steady speed of 30 m.p.h. In each case the resistances can be considered as constant and the same. Find the horse-power and the total resistance in lb.-wt.

5. Prove the formula $t = 2\pi\sqrt{(l/g)}$ for the period of small oscillations of a simple pendulum of length l.

A clock has a pendulum 3 ft. long and is losing 5 min. a day. Calculate approximately the adjustment which must be made to the length of the pendulum in order that the clock should give accurate time.

6. The governor of a gramophone consists essentially of a small weight on a light wire 2 in. long which rotates as a conical pendulum. The governor is geared to the turntable so that the angular velocity of the governor is twice that of the turntable. Find the semi-vertical angle of the conical pendulum when the turntable is rotating at 78 revs. per min.

7. A pair of steps consists of two similar ladders smoothly hinged together at the top and stands on a rough horizontal floor.

Each ladder is of weight w and length $2l$ and has its centre of gravity at its middle point. A weight W is placed on one of the ladders at a distance $\frac{1}{2}l$ from its lower end. Each ladder makes an angle θ with the vertical. Find the force of friction and the normal reaction of the ground on each ladder in terms of W, w and θ.

The steps are now gradually opened out, and slipping occurs when each ladder is inclined at an angle α to the vertical. Find which ladder slips first, and obtain the value of the coefficient of friction between a ladder and the ground in terms of W, w, and α.

8. A rod of weight W, whose centre of gravity divides its length in the ratio 2:1, lies in equilibrium inside a smooth hollow sphere. If the rod subtends an angle of 2α at the centre of the sphere and makes an angle θ with the horizontal, prove that $\tan\theta = \frac{1}{3}\tan\alpha$.

Find the reactions at the ends of the rod in terms of W and α.

9. Prove by integration that the centre of gravity of a uniform semicircular lamina of radius a is at a distance $4a/3\pi$ from the centre.

ACB is such a lamina with diameter AOB, and OC is the radius perpendicular to AB. A square portion $OPQR$ is cut out of the lamina, P being on OB and OP having length $\frac{1}{2}a$. Find the distances from OA and OC of the centre of gravity of the remaining portion.

Hence show that, if the remaining portion is suspended from A and hangs in equilibrium, the tangent of the angle made by AB with the vertical is just less than $\frac{1}{2}$.

10. A uniform cube of edge $4a$ and weight W stands on a rough horizontal plane. A gradually increasing force P is applied inwards at right angles to a face F of the cube at a point distant a vertically above the centre of that face. Prove that equilibrium will be broken by sliding or by tilting according as the coefficient

of friction between the cube and the plane is less than or greater than a certain value, and find this value.

If P has not reached a value for which equilibrium is broken, and $P=\frac{1}{2}W$, find how far from the face F of the cube the normal reaction acts.

Statistics

11. A rough rule is that the mean deviation μ and the standard deviation σ of a distribution are connected by the relation $\mu=\frac{4}{5}\sigma$.

Find whether this rule applies closely to the set of numbers 1, 2, 3, ... 7, 8, 9.

Find μ and σ for the set of numbers, 1, 2, 3, ..., $(2n-1)$.

[You may assume that

$$1^2+2^2+3^2+\ldots+n^2=\tfrac{1}{6}n(n+1)(2n+1).]$$

12. Explain what is meant by a *binomial distribution.*

A hen may be expected to lay five eggs a week. If six hens are kept, calculate the chance of getting (*a*) six eggs in one day, (*b*) four eggs in one day.

13. In the theory of samples drawn from a population, the proofs of two theorems are needed; first, that the mean of the means of the samples is the same as the mean of the population; secondly, that the standard error of these means is $\sigma/\sqrt{n}$, where σ is the standard deviation of the population and n is the sample size. Prove the *first* of these theorems.

A machine is intended to make insulators 3 cm. long. These are satisfactory provided their standard deviation is not more than 0·01 cm. As the machine gets older its accuracy is tested by finding the mean lengths of sample batches of 20 insulators. It is found that the standard deviation of the means of a number of batches is 0·005 cm. State, with reasons, what conclusion you draw about the condition of the machine.

14. Draw a graph, and on it fit in by eye the approximate regression lines of y on x and x on y for the following marks obtained by a set of 100 boys in a test in mathematics and a test in physics. $\bar{y}$ is the mean mark in physics of boys obtaining x marks in mathematics, and $\bar{x}$ is the mean mark in mathematics of boys obtaining y marks in physics.

Use your graph to find the coefficient of correlation between the two sets of marks. Interpret your answer.

x	1	2	3	4	5	6	7	8	9	10
$\bar{y}$	2·2	3·2	4·7	4·2	5·6	6·0	6·2	6·7	7·8	8·4
y	1	2	3	4	5	6	7	8	9	10
$\bar{x}$	3·0	3·3	4·0	4·2	5·2	5·3	6·0	7·2	7·2	7·8

MATHEMATICS 185

SCHOLARSHIP PAPER

PAPER IV

(*Three hours*)

Begin each answer on a fresh sheet of paper and arrange your answers in numerical order.

Mathematical tables and squared paper are provided.

1. Solve the simultaneous equations

$$x+2y-z=2,$$
$$2x-y+z=0,$$
$$\lambda x+5y-3(1+\lambda)z=4,$$

for the unknowns x, y, and z, where λ is a constant which is not zero.

Consider the case $\lambda=0$.

2. Show that if x is real the function

$$\frac{x^2}{(x-2)(x+3)}$$

cannot lie between the values 0 and 24/25. Sketch the graph of the function.

3. An arc AB of a circle of unit radius subtends an angle 2α at the centre O. The point G_0 is the centre of gravity of the arc AB. Prove that OG_0 equals $(\sin\alpha)/\alpha$.

The point P_0 bisects the arc AB and G_1 is the centre of gravity of the arc AP_0; P_1 bisects the arc AP_0 and G_2 is the centre of gravity of the arc AP_1; P_2 bisects the arc AP_1 and G_3 is the centre of gravity of the arc AP_2; and so on. Prove that

$$OG_0 = \cos\frac{\alpha}{2}.\cos\frac{\alpha}{2^2}.\cos\frac{\alpha}{2^3}\ldots\cos\frac{\alpha}{2^n}\,OG_n.$$

Hence, or otherwise, find the value which

$$\cos\frac{\alpha}{2}.\cos\frac{\alpha}{2^2}.\cos\frac{\alpha}{2^3}\ldots\cos\frac{\alpha}{2^n}$$

approaches as n tends to infinity.

4. Prove that

$$4\tan^{-1}\tfrac{1}{5} - \tan^{-1}\tfrac{1}{239} = \tan^{-1}\tfrac{120}{119} - \tan^{-1}\tfrac{1}{239} = \frac{\pi}{4}.$$

Using the expansion $\tan^{-1}x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots$, evaluate π to five decimal places.

5. Show that the locus of points of intersection of perpendicular normals to the parabola $y^2 = 4ax$ is

$$y^2 = a(x - 3a).$$

6. The coordinates of a point referred to rectangular axes are (x, y), and the coordinates of this point referred to a second set of axes with the same origin but turned through an angle α (measured from Ox towards Oy) relatively to the first set of axes are (x', y'). Prove that

$$x' = x\cos\alpha + y\sin\alpha, \quad y' = -x\sin\alpha + y\cos\alpha.$$

Write down equations expressing x and y in terms of x' and y'.

If the coordinates of two points P_1 and P_2, referred to the two sets of axes, are (x_1, y_1), etc., prove that

$$(x'_1 - x'_2)^2 + (y'_1 - y'_2)^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2,$$
$$x'_1 y'_2 - x'_2 y'_1 = x_1 y_2 - x_2 y_1,$$

and give the geometrical interpretation.

7. $f(x)$ is a function such that (i) $f(a) = a$ for a particular number a, and (ii) $f'(x)$ lies between 0 and 1 for values of x near a. A number x_1 is chosen near a, and numbers $x_2, x_3 \ldots$ are defined by

$$x_{n+1} = f(x_n) \quad (n = 1, 2, \ldots).$$

Illustrate this process graphically on a figure on which the curves $y = x$ and $y = f(x)$ are drawn, and show by means of your figure that x_n approaches the value a as n increases.

Find, correct to three decimal places, the solution near $x = 8$ of the equation

$$x = 10\log_{10}x - 1.$$

8. Find:

(i) $\int \log_e 3x\,dx$;

(ii) $\int \sin^4 x\,dx$;

(iii) $\int \frac{dx}{(1+e^x)(1+e^{-x})}$.

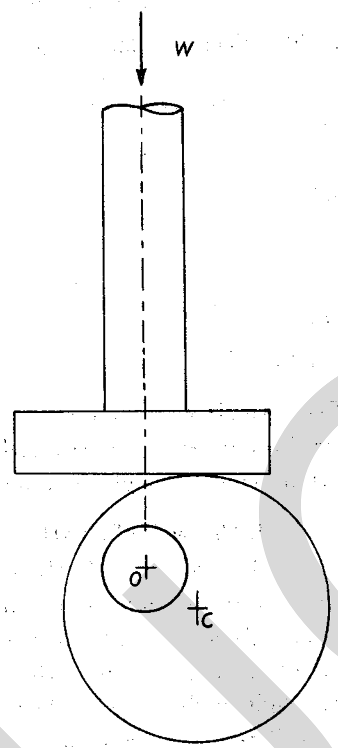
9. Find, from first principles, the least velocity required to project a particle over a barrier at a horizontal distance a from the point of projection and at a height b above it.

10. Two light rods OA, OB each of length a are free to rotate in a vertical plane about a point O. Particles of mass m are attached to the rods at A and B, and these points are joined by a light string of fixed length $a\sqrt{2}$. Initially the rods are at right angles and each makes an angle $\frac{1}{4}\pi$ with the upward vertical. The system is then slightly displaced. Show that the string remains in tension until the rods have turned through a right angle, and calculate the tension in the string and the thrust in each rod during this part of the motion in terms of the angle θ through which the system has turned.

11. A rod is moved in a smooth vertical guide by means of a cam mounted on a slowly rotating horizontal shaft. The cam has the form of a circle of radius $2a$ with its centre C at distance a

from the centre of rotation O. The rod presses down on the cam with a vertical force W, and the coefficient of friction between the cam and the rod is μ. Ignoring the weight of the cam and the friction in the bearings of the shaft, show that, if $\mu > 1/\sqrt{3}$, the torque required to turn the shaft is always positive and varies between

$$Wa\,[2\mu+\sqrt{(1+\mu^2)}] \quad \text{and} \quad Wa\,[2\mu-\sqrt{(1+\mu^2)}].$$



12. A sample of size n is drawn from a population in which a proportion p of individuals possess a certain characteristic and a proportion q do not (so that $p+q=1$). According to the theory of the binomial distribution the probability that the sample contains r individuals with the characteristic is

$$\frac{n!}{r!\,(n-r)!}\,p^r q^{n-r}.$$

Show that the mean value of r is np and that its standard deviation is $\sqrt{(npq)}$.

The following data refer to mortality rates in 1949:

	Live births	Deaths under one year
England and Wales	730,518	23,882
Leeds	8,407	254

Do these data provide any evidence for supposing that the death-rate in Leeds is not typical of that in the country as a whole?